The MOND limit from space-time scale invariance

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ABSTRACT

The MOND limit is shown to follow from a requirement of space-time scale invariance of the equations of motion for nonrelativistic, purely gravitational systems; i.e., invariance of the equations of motion under $(t, \mathbf{r}) \rightarrow (\lambda t, \lambda \mathbf{r})$ in the limit $a_0 \rightarrow \infty$. It is suggested that this should replace the definition of the MOND limit based on the low-acceleration behavior of a Newtonian-MOND interpolating function. In this way, the salient, deep-MOND results–asymptotically flat rotation curves, the mass-rotational-speed relation (baryonic Tully-Fisher relation), the Faber-Jackson relation, etc.–follow from a symmetry principle. For example, asymptotic flatness of rotation curves reflects the fact that radii change under scaling, while velocities do not. I then comment on the interpretation of the deep-MOND limit as one of "zero mass": Rest masses, whose presence obstructs scaling symmetry, become negligible compared to the "phantom", dynamical masses–those that some would attribute to dark matter. Unlike the former masses, the latter transform in a way that is consistent with the symmetry. Finally, I discuss the putative MOND-cosmology connection in light of another, previously known symmetry of the deep-MOND limit. In particular, it is suggested that MOND is related to the asymptotic de Sitter geometry of our universe. It is conjectured, for example, that in an exact de Sitter cosmos, deep-MOND physics would exactly apply to local systems. I also point out, in this connection, the possible relevance of a de Sitter-conformal-field-theory (dS/CFT) duality.

1. Introduction

MOND has been advanced as an alternative paradigm to Newtonian dynamics, whose original motivation was to explain the mass discrepancies in galactic systems without invoking dark matter (Milgrom 1983, for reviews see Sanders and McGaugh 2002, Bekenstein 2006, and Milgrom 2008). The paradigm is constructed on three premises: (i) There appears in non-relativistic physics a new constant, a_0, with the dimensions of acceleration. (ii) A correspondence principle is required that guarantees restoration of Newtonian physics when we formally take $a_0 \rightarrow 0$ in all the equations of motion (just as relativistic physics tends to classical physics when we formally take the speed of light $c \rightarrow \infty$). This matches the original introduction of MOND as a modification of Newtonian dynamics only when accelerations in the system are not much larger than a_0. (iii) Some constraints on the behavior of the theory in the opposite, deep-MOND limit. This limit is achieved by formally taking $a_0 \rightarrow \infty$ in the equations of motion.

Initially (Milgrom 1983 and onward) this deep-MOND limit was defined in terms of some MOND interpolating function of the acceleration, call it $\mu(a/a_0)$, which is required to have the limit $\mu(x) \approx x$ for $x \ll 1$, corresponding to $a_0 \rightarrow \infty$. This requirement is based solely on the axiom of MOND that isolated masses should have an asymptotically flat rotation curve, and so the interpolating function in question has to be related to rotation curves. Some effective MOND formulations (such as that of Bekenstein and Milgrom 1984) may indeed be defined by just one interpolating function. But in general the theory should not be defined by an interpolating function: The theory of relativity and quantum mechanics, which also constitute

departures from classical physics, do not hinge on a fundamental function that connects them smoothly to the classical regime. To be sure, interpolating functions have to appear in specific contexts of MOND just as they do in relativistic or quantum expressions[1]; but, these are not necessarily fundamental and may even have different asymptotic behaviors.

Clearly, it is desirable to define the MOND limit in a general way, without recourse to interpolating functions. Starting in Milgrom (2001), and more elaborately in Milgrom (2008), I defined the MOND limit for purely gravitational systems by the requirement that for $a_0 \to \infty$, the equations of motion are writable in a form where the constants a_0, G, and all masses in the problem, m_i, appear only in the products $m_i G a_0$. I also demonstrated there how this dictate, with the other two tenets of MOND, leads to many of the salient MOND predictions. While this requirement is operationally sound, it may appear artificial [see discussion around eq.(15) below].

Here I point out that the third MOND tenet, defining the deep-MOND limit, can be based on a symmetry requirement: A (nonrelativistic) MOND theory for a purely gravitational system has to becomes space-time scale invariant in the limit $a_0 \to \infty$; i.e., invariant under $(t, \mathbf{r}) \to (\lambda t, \lambda \mathbf{r})$ (leaving intact dimensioned constants of the system such as masses, G, and a_0–scaling is not a mere change of units).

This belated observation is important because (i) it may help put MOND on more sound footings, showing that it need not be imposed as an ad-hoc dictate of phenomenology; (ii) identifying an underlying symmetry may help extend MOND to non gravitational systems for which we have no phenomenological guidance, at present; (iii) identifying a partial symmetry valid in the deep-MOND regime may lead us to identify a larger symmetry in this regime, or to identifying a symmetry group for the full MOND theory, and this will help constrain the MOND theory itself; (iv) this description is more readily amenable to direct predictions.

There are several, possibly unrelated, facts that hint to the possibility that MOND is underlaid by a new symmetry, so that even as a non-relativistic theory it enjoys a space-time symmetry other than the Galilei group. For example, I showed in Milgrom (1994) that a MOND theory that is derived from an action and has the Newtonian and the deep-MOND limits as required, and enjoys Galilean invariance, cannot be local. So perhaps a local theory is possible with an underlying symmetry other than Galilei. In Milgrom (1997) I noted that the deep-MOND limit of the equation for the gravitational potential, in the formulation of Bekenstein and Milgrom (1984), is invariant to conformal space transformations. In Milgrom (2005) I commented on the superficial similarity between the MOND kinematics revolving around accelerations with the special relativistic kinematics revolving around velocities, with a_0^2 playing the role of $-c^2$, and with the new underlying symmetry having perhaps to do with de Sitter (hereafter dS) space-time[2] in which a_0 is a proxy for $c^2 \Lambda^{1/2}$.

On the other hand, the symmetry discussed here may be incidental, and specific to the purely gravitational case. This would be similar to the equations of motion of Newtonian gravity being invariant under $(t, \mathbf{r}) \to (\lambda t, \lambda^{2/3} \mathbf{r})$ (more on this in the discussion section), or to the problem involving harmonic forces being invariant under $(t, \mathbf{r}) \to (t, \lambda \mathbf{r})$. In these two cases the action governing the theories is not invariant,

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[1]Examples in relativity are the Lorentz factor, and various functions describing the behavior of particles near black holes, which tend to the classical form when $c \to \infty$. In the quantum-classical case, the black-body function, the expression for the specific heat of solids, or barrier transmission probabilities by tunneling are examples of interpolating functions that tend to the classical expression for $\hbar \to 0$.

[2]Even if the "dark energy" is a "cosmological constant", is not the sole contribution to the matter content of our universe. We thus do not live in an exact de Sitter space-time, and this might result in the breaking of the symmetries relevant to MOND.

but multiplied by a constant factor, under the corresponding scaling. This means that while the equations of motion are invariant, there are no conservation laws associated with the symmetry. If the scale invariance described here for the deep-MOND limit is of the same kind, and does not have fundamental underpinnings, it is still a useful tool, especially since MOND is inherently nonlinear, and is, otherwise, hardly amenable to deduction of exact results.

In section 2 I explain how the assumed space-time scaling invariance leads to the earlier formulations of the third tenet of MOND. In section 3 I show how scaling directly begets a variety of predictions. Section 4 describes the deep-MOND limit as a formal zero mass limit. In section 5 I discuss possible connections of MOND with cosmology and the interpretation of the deep-MOND limit. Section 6 is a discussion.

2. Space-time scale invariance and the deep-MOND limit

Consider a purely gravitational system in the deep-MOND regime. For convenience, I assume that the system is made of discrete masses m_i[3]. The assumption of pure gravity implies that the only physical constants that appear in the description of the system are G, a_0, and masses m_i. I do allow forces that constrain some of the masses to move on prescribed trajectories (such as keeping them at fixed positions) as such forces do not introduce additional dimensioned constants (the prescribed trajectories will always undergo scaling along with the dynamical ones). The problem we study consists of determining the trajectories of the gravitating particles, given some initial conditions. These are determined from equations of motion of the general form (combining dynamics with gravity)

$$\mathcal{F}_k[m_i, G, a_0, \mathbf{r}_i(t)] = 0, \quad k = 1, 2, 3, \dots \quad . \tag{1}$$

Here, $\mathbf{r}_i(t)$ stand for trajectories of the particles, including those prescribed, and $\mathcal{F}_k$ are general functionals of the trajectories, so they can contain any number of derivatives of $\mathbf{r}_i(t)$, or be non-local functionals of them. The statement of the symmetry required in the deep-MOND limit is then: In the limit $a_0 \to \infty$, if $\mathbf{r}_i(t)$ constitute an allowed configuration of the particle motions, then so do $\hat{\mathbf{r}}_i(\hat{t}) = \lambda \mathbf{r}_i(t) = \lambda \mathbf{r}_i(\hat{t}/\lambda)$ (corresponding to $\hat{\mathbf{r}}_i = \lambda \mathbf{r}_i$, and $\hat{t} = \lambda t$), with the appropriate scaling of the initial conditions. In other words, if eq.(1) holds for some $\mathbf{r}_i(t)$ then it is also true that

$$\mathcal{F}_k[m_i, G, a_0, \lambda \mathbf{r}_i(t/\lambda)] = 0, \quad k = 1, 2, 3, \dots \quad . \tag{2}$$

Note that the constants and the masses remain intact[4].

The equations of motion may also involve gravitational fields that mediate gravity, with their own transformation properties under scaling that insure the invariance of the equations of motion. But, I concentrate on the observable motion of the masses $\mathbf{r}(t)$, and on what the symmetry implies for them. (One may think,

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[3]For a truly point mass, the field near the mass is not in the deep MOND regime; so, when we speak of masses we have to view them as larger than their MOND transition radius $r_t \equiv (MGg/a_0)^{1/2}$, or consider only the regions outside r_t for all masses. For reference, note that the transition radius of a proton is $\sim 10^{-12}$ cm.

[4]Sometimes we describe part of the system as a fluid, characterized by a density and velocity fields, $\rho(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$, respectively (instead of by masses and their positions). These fields then transform under scaling as $\rho(\mathbf{r}, t) \to \lambda^{-3} \rho(\mathbf{r}/\lambda, t/\lambda)$ and $\mathbf{v}(\mathbf{r}, t) \to \mathbf{v}(\mathbf{r}/\lambda, t/\lambda)$

for example, of equations of motion from which the gravitational fields have been eliminated by expressing them in terms of the particle trajectories and substituting them back in the equations of motion.)

To make contact with the earlier definition of the deep-MOND limit, note that from dimensional considerations alone, we are free to multiply all quantities with dimensions of mass, lengths, and time by arbitrary, possibly dimensioned, constants, α, η, and ξ, without impairing the validity of an equation. Equation (1) can thus be written as

$$\mathcal{F}_k[\alpha m_i, \eta^3 \xi^{-2} \alpha^{-1} G, \eta \xi^{-2} a_0, \eta \mathbf{r}_i(t/\xi)] = 0. \tag{3}$$

Take now $\xi = \eta = a_0$ and $\alpha = Ga_0$, and impose the symmetry to get

$$\mathcal{F}_k[(m_i Ga_0, 1, 1, \mathbf{r}_i(t)] = 0, \tag{4}$$

which proves that the equations governing the motion of particles can be brought to a form in which m_i, G, a_0 appear only in the products $m_i Ga_0$. This is the formulation of the third MOND tenet as given in Milgrom (2008), which is now seen to follow from scale invariance[5]. Clearly the opposite is also true, as stated in Milgrom (2008). The above result simply reflects the fact that any combination of M, G, and a_0 whose value is invariant under joint scaling of the length and time units must be a function of MGa_0.

As an example, the very first description of the deep-MOND limit (Milgrom 1983) giving the acceleration of a test particle at a distance R from a point mass M as:

$$(a/a_0)\mathbf{a} = -MG\mathbf{R}/R^3 \tag{5}$$

is clearly scale invariant in the above sense.

So far I have only discussed the equations of motion, which are more directly related to phenomenology. What can be said about the effect of scaling on an underlying action? To obtain scale invariance of the equations of motion it is sufficient that the action is multiplied by a constant under scaling. However, if the action is not truly invariant, only multiplied by a constant, the symmetry does not imply a conservation law. We don't know what action will govern the final MOND theory; but start, as an example, with the Newtonian action governing a nonrelativistic system of gravitating point masses:

$$S = -\frac{1}{8\pi G} \int dt d^3 r \, (\vec{\nabla}\phi)^2 - \sum_i m_i \int dt \phi[\mathbf{r}_i(t)] + \sum_i m_i \int dt \, v_i^2(t)/2. \tag{6}$$

Its different terms cannot transform in the same way under scaling: Comparing the last two terms, we see that for them to transform in the same way, ϕ would have to have zero dimension under scaling (i.e., transform as $\phi(\mathbf{r}, t) \rightarrow \phi(\mathbf{r}/\lambda, t/\lambda)$; but, then, the first term does not transform like the other two.

One way to effect uniform transformation properties for the three terms, with the help of a_0, is to modify the first term[6] replacing $(\vec{\nabla}\phi)^2$ by another Lagrangian density that has three powers of length

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[5]Note that this conclusion does not change if the theory also involves physical constants with the dimensions of velocity (such as the speed of light), whose values do not change under scaling. But our subsequent results do assume the irrelevance of such additional constants.

[6]I call such modifications "modified gravity" because they modify the gravitational field, leaving Newton's second law intact.

in the denominator[7]. For example it can be proportional to $|\vec{\nabla}\phi|^3$, or if we allow higher derivatives of the potential, $(\Delta\phi)^{3/2}$, $(\phi_{,i,j}\phi_{,k}\phi_{,l}\delta^{ik}\delta^{jl})^{3/4}$, etc.. However, since we have only a_0 at our disposal as an additional dimensioned constant we can only form modified Lagrangian densities involving the first derivatives of ϕ (if we allow only derivatives of ϕ, not ϕ itself, to appear). Our assumed scale invariance thus leads to a unique Lagrangian for the deep-MOND limit in this single potential case; i.e., $|\vec{\nabla}\phi|^3/a_0$, which is indeed the deep-MOND limit of the Bekenstein and Milgrom (1984) formulation. The field equation for the gravitational potential then becomes a modification of the Poisson equation with the Laplacian replaced by the 3-Laplacian $\vec{\nabla}\cdot(|\vec{\nabla}\phi|\vec{\nabla}\phi)$.

Interestingly, I showed in Milgrom (1997) that the deep-MOND limit of the Bekenstein and Milgrom (1984) formulation for the field equation for the potential is, in fact, invariant under the whole group of conformal *space* transformation (which includes space scaling)[8]. The above deep-MOND action is not invariant, but is multiplied by a constant under scaling.

Another option to construct a MOND action is to modify the kinetic action of particles–so called "modified inertia" (Milgrom 1994): Replace the last term in expression (6) by a kinetic action of the form

$$S_k = M\tilde{S}_k[a_0, q_1, q_2, ..., \mathbf{r}_1(t), \mathbf{r}_2(t), ...], \tag{7}$$

where $M = \sum_i m_i$, and, on dimensional grounds, $\tilde{S}_k$, which has dimensions of length^2/time, can only be a function of the mass ratios $q_i = m_i/M$, in addition to its being a function of a_0 and a functional of the particle trajectories $\mathbf{r}_i(t)$. In order to insure universality of free fall we require that $\tilde{S}_k$ is symmetric under interchange of particle indices (a possible form is $\tilde{S}_k = \sum_i q_i\tilde{s}_k[a_0, \mathbf{r}_i(t)]$). The first two terms in expression (6) have the same transformation properties if ϕ is given dimension -1 under scaling [by decree, complementing the transformation rule of t and $\mathbf{r}$ by $\phi(\mathbf{r}, t) \rightarrow \lambda^{-1}\phi(\mathbf{r}/\lambda, t/\lambda)$]. Then, to have the same transformation properties as the first two we have to have in the limit $a_0 \rightarrow \infty$,

$$\tilde{S}_k \rightarrow a_0^{-1}\sigma[q_1, q_2, ..., \mathbf{r}_1(t), \mathbf{r}_2(t), ...], \tag{8}$$

where σ is a functional of the trajectories (and function of the mass ratios) with dimensions of length^3/time^3. Multiplying the action in eq.(6) by Ga_0^2 and absorbing one power of a_0 in the definition of ϕ, defining $\psi = a_0\phi$, we get, in the limit, an action in which only m_iGa_0 appear:

$$S = -\frac{1}{8\pi}\int dt d^3r \, (\vec{\nabla}\psi)^2 - \sum_i m_iGa_0\int dt\psi[\mathbf{r}_i(t)] + MGa_0\sigma[q_1, q_2, ..., \mathbf{r}_1(t), \mathbf{r}_2(t), ...]. \tag{9}$$

As in the Newtonian case, the potential can be eliminated from the action: First note that for solutions of the equations of motion the first term in the action is $-1/2$ times the second, which, in turn, can be explicitly expressed in terms of the positions to give

$$S = -\int dt\Phi[\mathbf{r}_i(t)] + MGa_0\sigma, \tag{10}$$

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[7]In our nonrelativistic case, time derivatives of the potential do not appear, as the potential adjusts itself instantaneously to motion of the masses; so, the field equations hold time by time.

[8]This turns out to be the case in all the above mentioned scale invariant gravitational Lagrangian containing higher derivatives of the potential, not only the MOND case. It is well known that scale invariance, together with rotational invariance, implies conformal invariance in a large class of theories. Our theory is not invariant to rotations in space-time–i.e., is not Lorentz invariant; so, the space-time scale invariance does not imply full space-time conformal invariance, but perhaps a generalization of it will.

where the potential energy

$$\Phi = -\sum_{i<j} \frac{m_i m_j (Ga_0)^2}{|\mathbf{r}_i(t) - \mathbf{r}_j(t)|}. \tag{11}$$

This "modified-inertia", deep-MOND action in expressions (9) or (10) is truly invariant under scaling $(t, \mathbf{r}, \psi) \to (\lambda t, \lambda \mathbf{r}, \lambda^{-1}\psi)$. For reasons explained in Milgrom (1994) we ought to concentrate on nonlocal forms of σ, in which case no conservation law is implied. For completeness I mention that for local kinetic actions, the symmetry would imply a conservation law. For example, if the kinetic action can be written in terms of a kinetic Lagrangian depending on derivatives up to order n:

$$MGa_0\sigma = \sum_i \int dt L[\dot{\mathbf{r}}_i, \ddot{\mathbf{r}}_i, ..., \mathbf{r}_i^{(n)}], \tag{12}$$

the conserved quantity can be shown to be

$$Q \equiv -Ht + \sum_i \sum_{m=1}^{n} \mathbf{r}_i^{(m-1)} \cdot \mathbf{p}_i^m. \tag{13}$$

Here, the energy H and the momenta $\mathbf{p}_i^m$ are those introduced by Ostrogradski, expressions for which are given, e.g., in Milgrom (1994). (Invariance to time translations implies the conservation of H, and to space translations the conservation of $\mathbf{P}^1 = \sum_i \mathbf{p}_i^1$.)

The division into "modified gravity" and "modified inertia" is not exhaustive, and, not even always well defined, as we can devise simultaneous modifications of all terms in the action. Perhaps a more appropriate distinction is between modifications that are "modified gravity" and those that are not. In the former, nonrelativistic gravity is still described by a potential, ϕ, in which all masses still follow $\mathbf{a} = -\vec{\nabla}\phi$, but the potential is determined by an equation other than Poisson's. In the latter class of theories this is not so, and the above "modified inertia" action is an example of such a theory.

For the relativistic case we might call "modified gravity" a theory that can be cast as a metric one, with the standard coupling of matter degrees of freedom to the metric (implying geodesic motion). It departs from General Relativity in that the metric is not determined from the Einstein equations. The nonrelativistic limit of such theories is then a "modified gravity" theory in the above sense.

3. Predicted phenomenology

3.1. Asymptotic rotation speeds

There are several important implications for gravitational systems in the deep-MOND regime that can be deduced directly from the assumed scaling invariance. In the first place, it tells us that if $\mathbf{r}(t)$ is a trajectory of a point body in a configuration of point masses[9] m_i at positions $\mathbf{r}_i(t)$ (which can be taken as fixed, for example), then $\hat{\mathbf{r}}(t) = \lambda\mathbf{r}(t/\lambda)$ is a trajectory for the configuration where m_i are at $\lambda\mathbf{r}_i(t/\lambda)$, and the velocities on that trajectory are $\hat{\mathbf{V}}(t) = \mathbf{V}(t/\lambda)$. It follows from this, for example, that the rotational speed in an orbit around an isolated mass becomes independent of the size of the orbit in the large size limit:

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[9]Under scaling, an extended mass changes its size and density such that the total mass remains the same. A point mass remains a point mass of the same value.

Under scaling the orbit changes its size, but the velocities do not change. The extent of the attracting mass also scales, but for large orbital radii, the size of the central mass becomes immaterial and does not affect the motion[10]. This leads to the asymptotic flatness of rotation curves of isolated galaxies. In the present formulation of the MOND limit, this cornerstone axiom of MOND is thus simply a reflection of the fact that velocities do not change under space-time scaling, while distances do.

Accelerations scale as λ^{-1}; so, in the deep-MOND limit, when a physical orbit is scaled up to another the accelerations scale as the inverse of the orbit's size. This scaling of the acceleration does not, in general, relate accelerations at different point on the same orbit. However, in the class of theories constituting "modified gravity", namely, theories in which the gravitational field is modified, and the acceleration depends only on position, the above scaling does tell us that the acceleration of a test mass in the field of single point mass decreases as the inverse of the distance from that mass[11]. It also tells us, more generally, that the gravitational force between two point masses must decrease as the inverse of the distance between them.

3.2. Mass-velocity relations

A change of the time units by a factor ξ in eq.(4) gives

$$\mathcal{F}_k[(\xi^{-4}m_i G a_0, 1, 1, \mathbf{r}_i(t/\xi)] = 0, \tag{14}$$

which tells us how orbital characteristics change with scaling of the masses.

Before discussing the implications of this relation I derive it in another way, which gives an added insight. The appearance of dimensioned constants in physics is, many times, artificial, and results from our insistence on inventing new units for quantities that can be measured in existing ones. But such constants may, actually, be useful as bookkeeping tools, as in the examples above, and also in helping us implement certain limits such as the nonrelativistic limit, or, as here, the MOND limit. Alternatively, we may work in units in which $a_0 = G = 1$ are dimensionless[12]. Then, lengths have dimensions of time squared, and masses have dimensions of time to the fourth power[13]. The MOND equations of motion (1) are now of the form

$$\hat{\mathcal{F}}_k[m_i, \mathbf{r}_i(t)] = 0, \quad k = 1, 2, 3, \dots \; , \tag{15}$$

where we cannot even describe the MOND limit in terms of how the constants appear. Scale invariance can be implemented as follows: under change the units of time by a dimensionless factor ξ, eq.(15) becomes

$$\hat{\mathcal{F}}_k[\xi^{-4}m_i, \xi^{-2}\mathbf{r}_i(\xi t)] = 0, \quad k = 1, 2, 3, \dots \; . \tag{16}$$

Now apply the scale invariance (valid in the deep-MOND limit), which tells us that $\mathbf{r}_i(\xi t)$ in eq.(16) can be replaced by $\lambda \mathbf{r}_i(\xi t/\lambda)$. Choosing $\lambda = \xi^2$ gives

$$\hat{\mathcal{F}}_k[\xi^{-4}m_i, \mathbf{r}_i(t/\xi)] = 0, \quad k = 1, 2, 3, \dots \; , \tag{17}$$

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[10]I assume that the effect of a mass becomes independent of its internal structure, including its extent, when this extent becomes very small. Otherwise the concept of point mass loses its meaning anyway.

[11]This also follows from the statement, made above, that in modified gravity the potential has to have zero scaling dimension; this in turn, implies that it has to be logarithmic in the distance in the deep-MOND limit.

[12]Defining the MOND and Newtonian limits then becomes somewhat more elaborate: we have to identify all quantities with the role of acceleration, and require that they are much smaller than 1, or much larger than 1, respectively.

[13]Under change of the units of time we then have: $t \to \xi^{-1}t$, $\mathbf{r} \to \xi^{-2}\mathbf{r}$, $m \to \xi^{-4}$; but under scaling we still have $t \to \lambda t$, $\mathbf{r} \to \lambda \mathbf{r}$, $m \to m$.

which has the same content as eq.(14). Thus, if we have a solution $\mathbf{r}_i(t)$ for a given choice of the masses, then $\mathbf{r}_i(t/\xi)$ is a solution for a system where all the masses are multiplied by ξ^{-4}. The orbits remain the same in space but the bodies traverse them in times multiplied by a factor ξ, speeds that are multiplied by ξ^{-1}, and accelerations that are multiplied by ξ^{-2} (for the appropriately scaled initial conditions). More generally, choosing $\lambda = \xi^\alpha$ gives

$$\hat{\mathcal{F}}_k[\xi^{-4}m_i, \xi^{\alpha-2}\mathbf{r}_i(t\xi^{1-\alpha})] = 0, \quad k = 1, 2, 3, \dots \; . \tag{18}$$

The velocities for all these solutions still scale as the fourth root of the masses. If the theory is such that the trajectory $\mathbf{r}(t)$ of a test particle does not depend on its mass–as required by the universality of free fall, and as I assume all along–we do not have to scale the mass of test particles.

From this scaling with mass follows the MOND mass-asymptotic-speed relation (underlying the baryonic Tully-Fisher relation)

$$V_\infty^4 = MGa_0. \tag{19}$$

This corollary also tells us that a MOND theory must be nonlinear: Since accelerations here scale as the square root of the mass (unlike the Newtonian scaling with mass) the acceleration, light bending, etc., that are produced by several masses is not the sum of those produced by the individual masses.

3.3. Virial relations

This scaling of velocities with mass also tightly constrains the form of the deep-MOND virial relation: Consider some measure of the mean squared velocity, $\langle V^2 \rangle$, in a stationary, many-body, self-gravitating, bound system of total mass M. For a system deep in the MOND regime the above scaling tells us that

$$\langle V^2 \rangle^2 = MGa_0 Q, \tag{20}$$

where Q is a function of dimensionless attributes of the system (such as mass ratios, geometrical factors, velocity anisotropies, etc.), which, in particular, does not depend on the overall size of the system. The function Q also depends on the particular formulation of MOND at hand. For example, for the formulation of Bekenstein and Milgrom (1984), the space, conformal invariance of the deep-MOND limit was shown (Milgrom 1997) to lead to the following general, exact result: If $\langle V^2 \rangle$ is the mass weighted, mean square velocity $\langle V^2 \rangle = \sum m_i V_i^2/M$, then

$$Q = \frac{4}{9}(1 - \sum_i q_i^{3/2})^2, \tag{21}$$

which depends only on mass ratios in the system, q_i, and becomes a number, $Q = 4/9$, when all masses are small compared with the total mass $[Q = (4/9)(1 - N^{-1/2})^2$ for a system of N equal mass bodies]. This scaling law also underlies the Faber-Jackson relation for low acceleration systems. As explained in Milgrom (2008), with the help of the other MOND tenets, this extends to a predicted Faber-Jackson correlation for spherical systems in general, provided they can be approximately described as isothermal spheres.

I note in passing that the above virial relation as applied to thin disc galaxies in the deep-MOND regime reads

$$\langle V^2 \rangle^2 = \frac{4}{9}MGa_0, \tag{22}$$

where $\langle V^2 \rangle \equiv \int_0^\infty 2\pi r \Sigma(r) V^2(r) dr$, with $\Sigma(r)$ and $V(r)$ are, respectively, the normalized mass surface density $(2\pi \int r\Sigma(r)dr = 1)$, and the rotational speed in the disk. This resembles the mass velocity relation (19), but, in fact, it is a rather different, deep-MOND result that follows from the same scaling property: it involves the rms velocity, not the asymptotic one, it is specific to the Bekenstein & Milgrom (1984) formulation, and it applies only to deep-MOND galaxies unlike relation (19), which is universal.

Similar, but different, relations are expected in other MOND formulations. For example, the whole class of modified-inertia formulation of MOND (Milgrom 1994) predict the following virial relation for disk galaxies: Define a normalized, effective surface density of the disk, $\Sigma_*(r)$, such that the Newtonian acceleration is given by[14]

$$g_N(r) = \frac{V^2(r)}{r} = \frac{MG}{r^2} \int_0^r 2\pi r' \Sigma_*(r') dr', \tag{23}$$

and define now the rms rotational velocity weighted with $\Sigma_*(r)$ instead of $\Sigma(r)$. Then the relation

$$\langle V^2 \rangle_*^2 = \frac{4}{9} MGa_0 \tag{24}$$

is predicted by all modified inertia formulations of MOND for galaxies in the deep-MOND regime. This can be derived straightforwardly from the fact that in such theories we have for circular orbits in the deep-MOND limit $V^4(r)/r^2 a_0 = -d\phi/dr$, where $\phi(r)$ is the Newtonian potential in the disk.

With accurate enough data we can directly test these two classes of theories without having to calculate rotational speeds, which in the case of the nonlinear Bekenstein & Milgrom (1984) formulation is rather demanding (all the quantities appearing in the relations are directly observable).

More generally, scale invariance implies that for very low acceleration galaxies we have the following scaling property of the rotation curve. If a disk galaxy with a low surface density $\Sigma(r)$ has a rotation curve $V(r)$, then a disk with surface density $\alpha\Sigma(\beta r)$ has a rotation curve $\alpha^{1/4}\beta^{-1/2}V(\beta r)$ (instead of the Newtonian scaling $\alpha^{1/2}\beta^{-1/2}V(\beta r)$) provided both disks are in the deep-MOND regime.

3.4. The external-field effect

Sometimes one wants to describe a small, self gravitating system that is itself falling in a field of a mass external to it. A case in point is a globular cluster, or a dwarf spheroidal galaxy, falling in the field of a mother galaxy. When the extent of the system is small compared with the scale over which the external field varies, and the intrinsic time scales are much shorter than the characteristic fall time, we may describe the small system, approximately, as embedded in a constant (in space and time) acceleration field of the external body. The non-linearity of MOND leads, generically, to palpable effects of the external field on the internal motions in the system. This is known as the external-field effect (EFE) in MOND (Milgrom 1983). The system is controlled by the same MOND theory with boundary conditions of a constant acceleration field at infinity, in the case of modified gravity, or a system of equations transformed to a constantly accelerated frame, otherwise. The details do not concern us here; all we reckon with is the fact that in some way a new "constant" is added to the MOND theory: $\mathbf{g}_0$, the vector of external acceleration field. The equations of motion (1) can now be written symbolically:

$$\mathcal{F}_k[m_i, G, a_0, \mathbf{g}_0, \mathbf{r}_i(t)] = 0, \quad k = 1, 2, 3, \dots \ . \tag{25}$$

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[14]Σ_* is a sort of "spherical equivalent" surface density: If $\rho(r)$ is the spherical density distribution that gives the same Newtonian acceleration as the disk, then $\Sigma_* = 2r\rho/M$.

Note that rotational symmetry is now broken. While $\mathbf{g}_0$ is a constant of the effective theory thus obtained, unlike a_0, it does transform under scaling. The reason is that scale invariance holds only if we apply scaling to the whole system, including the masses that give rise to $\mathbf{g}_0$, and this will scale $\mathbf{g}_0 \to \lambda^{-1}\mathbf{g}_0$. Let us again work in units in which $G = a_0 = 1$ are dimensionless; so the equations of motion are written[15]

$$\mathcal{F}_k[m_i, g_0, \mathbf{r}_i(t)] = 0, \quad k = 1, 2, 3, \dots \quad , \tag{26}$$

where I also suppressed the dependence on the direction of $\mathbf{g}_0$. Now apply a change in the time units by a factor ξ, and scaling by a factor $\lambda = \xi^{-1}$. Scale invariance then tells us that the equations of motion

$$\mathcal{F}_k[\xi^4 m_i, \xi g_0, \xi \mathbf{r}_i(t)] = 0, \quad k = 1, 2, 3, \dots \quad , \tag{27}$$

are also satisfied. Define the quantities $\tilde{\mathbf{r}}_i(t) = \mathbf{r}_i(t)/g_0$, $\tilde{m}_i = m_i/g_0^4$, and we have that

$$\mathcal{F}_k[\tilde{m}_i, \xi g_0, \tilde{\mathbf{r}}_i(t)] = 0, \quad k = 1, 2, 3, \dots \quad . \tag{28}$$

Namely, if $\tilde{\mathbf{r}}_i(t)$ are a solution for a system with given $\tilde{m}_i$ and g_0, then it is also a solution for the value ξg_0, with arbitrary ξ, and the same $\tilde{m}_i$. In other words, in the equations expressed in terms of $\tilde{m}_i$ and $\tilde{\mathbf{r}}_i(t)$, g_0 disappears. Scale invariance thus tells us that the equations of motion can be cast in the form

$$\mathcal{F}_k[\tilde{m}_i, \tilde{\mathbf{r}}_i(t)] = 0, \quad k = 1, 2, 3, \dots \tag{29}$$

(the dependence on the direction of $\mathbf{g}_0$ is still there). In our units $\mathbf{r}_i$, as well as $\tilde{\mathbf{r}}_i$, have dimensions of time^2, while m_i and $\tilde{m}_i$ have dimensions of time^4. The second time derivatives of $\tilde{\mathbf{r}}_i$ (accelerations normalized by g_0) are thus dimensionless as are quantities of the construction $\tilde{m}/\tilde{r}^2$. We can thus write schematically an expression that describes the general scaling of accelerations in the system with masses, size, and g_0:

$$\tilde{a} = f(\tilde{m}/\tilde{r}^2). \tag{30}$$

Putting back the dimensioned constants G and a_0 we can write this as

$$a = g_0 f\left(\frac{mGa_0}{g_0^2 r^2}\right). \tag{31}$$

If it where not for scale invariance, a/g_0 could also depend on another dimensionless variable such as g_0/a_0, or other equivalent ones, but the argument of f in eq.(31) is the only combination that is both dimensionless and scale invariant (as is a/g_0). Note that eq.(31), while quite restrictive, still leaves much latitude for the effects of the external field, as f is not constrained, and can vary from theory to theory. For example, $f(x) \propto x$ describes the EFE in the original discussion of the effect in Milgrom (1983), and in the formulation of Bekenstein and Milgrom (1984). It gives a quasi-Newtonian behavior: $a \propto MG_e/r^2$, with an effective $G_e = Ga_0/g_0$. This is only the general scaling law; details-such as the angular dependence around the direction of $\mathbf{g}_0$–differ in the two theories. Scaling alone does not even imply an EFE, as $f(x) \propto x^{1/2}$ corresponds to a theory with no EFE, and gives the standard scaling for the deep-MOND limit in an isolated system. Note that the meaning of $\mathbf{g}_0$ as an external field has not played a role in the considerations above. All that was assumed is that the system is characterized by an additional constant with units and scaling dimensions of an acceleration.

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[15]I use $\mathcal{F}_k$ in all the equations that follow even though their meaning changes somewhat from equation to equation.

4.   The deep-MOND limit as a zero-rest-mass limit

It is well known that the appearance of finite rest masses in a theory is an obstacle to scale and conformal invariance. In the case of MOND the appearance of the new constant a_0, is potentially a further hinderance, as it scales as an inverse mass. The reason is briefly this: MG has dimensions of $V^2 r$; so if all masses were dynamical (as opposed to rest masses, which are constants of the theory), they would transform like length under scaling, instead of being considered untouchable constants: dimensional balance would have then insured that masses do not obstruct the symmetry. When considering conformal invariance under a change in metric $g_{\mu\nu} \to \kappa^2(x) g_{\mu\nu}$, the corresponding requirement is that rest masses transform as $M \to \kappa^{-1}(x)M$, as a necessary condition for the symmetry to hold (see, e.g., Fulton et al. 1962, Bekenstein and Meisels 1980). But since rest masses (and a_0) are generally considered "constants of nature", one is loath to have them transform in these ways, in which case one has to forgo the symmetry in theories with rest masses. However, since in the deep-MOND limit we only have products $m_i a_0$ appearing, they do not obstruct the symmetry. As I discuss below, there are other, "dynamical", masses that appear in MOND and that are not multiplied by a_0. These, however, automatically transform appropriately for the symmetry to remain unimpaired.

The deep-MOND limit may also be viewed as a limit of zero rest masses: Since in this limit rest masses appear only as $m_i a_0$–constants that control the dynamics of the system and stay fixed when taking the formal limit $a_0 \to \infty$–it follows that the same formal limit corresponds to $m_i \to 0$. This does not mean, of course, that the effects of gravity vanish in the limit. They are taken over by the "phantom" masses: the quantities with the dimensions of mass that govern the motions, and that would be attributed to dark matter by a dark matter advocate. In the deep-MOND limit, the rest masses become negligible compared to the "phantom masses", hence the formal limit of zero rest masses.

In nonrelativistic, modified gravity theories the "phantom" masses are easily envisaged. Let ϕ be the MOND potential produced by a rest mass distribution $\rho(\mathbf{r})$. The same potential will be produced in Newtonian gravity by a "dynamical" density

$$\rho_d \equiv (4\pi G)^{-1} \Delta\phi, \tag{32}$$

and the difference $\rho_* = \rho_d - \rho$, the so called "phantom" density, would be attributed to "dark matter" by those who prefer it[16]. In theories that cannot be described as "modified gravity" there is no unique distribution of "phantom matter" in a given system, but dynamics can still be described as being effectively controlled by quantities with the dimensions of mass. All such quantities have to scale as $GM_* \sim (MGa_0)^{1/2}\ell$, where M is a rest mass parameter, and ℓ is some characteristic length of the system, or of the motion, such as system size or orbital size. Given that rest masses can appear only in the combination MGa_0, this is the only way to construct a mass from the quantities available. Such mass quantities have two important properties: they remain finite in the deep-MOND limit , and they scale as length, so they do not obstruct scale invariance.

As an example, consider the MOND acceleration, say on a circular orbit, at a distance r from a point mass, M, which is given by

$$a = \frac{(M + M_*)G}{r^2}, \tag{33}$$

where M_* is just a parametrization of the departure from the Newtonian acceleration given by the first term. For small radii M_* goes to zero; for radii $r \gg r_t$–with $r_t = (MG/a_0)^{1/2}$ being the transition radius–M_*

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[16]There is no guarantee that ρ_* satisfies the standard positivity conditions; in fact, there are examples to the contrary.

dominates and approaches the value $M_* \approx Mr/r_t \gg M$, which indeed scales as described above. Equation (33) is not scale invariant for finite M, but becomes so in the deep-MOND limit, where $M/M* \to 0$.

To summarize, starting with a MOND theory that is characterized by the constants Gm_i and a_0, express these parameters by $q_i \equiv m_i/M$, $S \equiv MGa_0$, and M ($M = \sum m_i$, so $\sum q_i = 1$). The MOND limit is gotten by substituting $m_i = 0$, everywhere, with q_i and S fixed.

5. Possible cosmological connections

There is a well known numerical proximity between a_0 and cosmological acceleration scales. Observations suggest two such potentially relevant scales, one associated with the expansion rate $a_H \equiv cH_0$, where H_0 is the Hubble constant, and another associated with the measured "cosmological constant", $\hat{a}_0 \equiv c(\Lambda/3)^{1/2}$ (Λ here having units of t^{-2}). It so happens that these two are very near in value, in the present cosmological epoch (the "cosmological coincidence"), as expressed by the fact that $\Omega_\Lambda \equiv \Lambda/3H_0^2$ is today near 1. Since a_0 is a very prominent, universal, measured signature of galactic dynamics, it should be viewed as another mysterious coincidence (e.g., Milgrom 1983, 1989) that

$$\bar{a}_0 \equiv 2\pi a_0 \approx a_H \approx \hat{a}_0. \tag{34}$$

This may bespeak a physical connection between MOND and the entity behind Λ and/or the expansion rate.

Instead of working with the constant a_0 we could work with a mass constant $M_0 \equiv c^4/Ga_0$. From relation (34), we have today $M_0 \approx 2\pi M_U$, where $M_U \equiv c^3G^{-1}(\Lambda/3)^{-1/2} \approx c^3G^{-1}H_0^{-1}$ is the characteristic energy in the universe within the horizon today. Our findings above are then tantamount to having in the deep-MOND limit all masses appear in the equations of motion only in the ratios m_i/M_0. For example, the mass-velocity relation predicted by MOND eq.(19) takes the form

$$(V_\infty/c)^4 = M/M_0. \tag{35}$$

We need a dynamical theory that connects MOND with cosmology to deduce, among other things, which of the cosmological acceleration scales, if any of the two, is to be identified with a_0. In fact, a_0 could depend in a more complicated way on the cosmological state. For example, we can write heuristically

$$\bar{a}_0 = a_H \mathcal{A}(\hat{a}_0/a_H). \tag{36}$$

Pinpointing such a dependence would have great consequences regarding the cosmological time variations of a_0, with obvious ramifications for both cosmology and galaxy dynamics, as I discussed elsewhere. Examples of possible cosmological variations of a_0 in specific, relativistic MOND formulations was discussed by Sanders (2005), and by Bekenstein and Sagi (2008). The potential for constraining such variations observationally was discussed recently in Milgrom (2008), and in Limbach et al. (2008). One obvious possibility that has been considered (and also used in structure formation simulations) is that a_0 is proportional to $\hat{a}_0$, and that the "dark energy" is a veritable constant. In this case, a_0 would not vary with cosmic time. Other possibilities exist, of course; for example, the interesting one I discuss below.

The following facts might, together, prove very relevant to the MOND-cosmology connection:

1. If the "dark energy" is a "cosmological constant", it will become increasingly dominant (over matter) in the future, with $\Omega_\Lambda \to 1$. The space-time we live in will then approach an exact de Sitter geometry (characterized by $\Omega_\Lambda = 1$) in the cosmological future.

2. The symmetry (isometry) group of such an exact, 4-dimensional, de Sitter space-time, dS^4, is isomorphic (equivalent) to the group of conformal transformations in 3-dimensional Euclidean space[17]; it is also the conformal group acting on the 3-dimensional Euclidean sphere. The two groups are ten dimensional. In the former group–analogous (but not equivalent) to the Poincaré group in a flat space-time–the transformations can be represented by the ten rotations about the origin in the 5-dimensional, flat, Minkowski space in which a dS^4 can be embedded. They can be described as corresponding, in the limit of an infinite radius of the dS^4 (or $\Lambda = 0$ with no matter), to three space rotations, three space translations, one time translation, and three Lorentz boosts. In the 3-dimensional Euclidean space, time clearly disappears, and the equivalent, conformal group is generated by the three (space) rotations, three translations, one dilatation, and three so-called proper conformal transformations (as it were, the four operations involving time: time translations and boosts, metamorphose into space dilations and proper conformal transformations in space).

3. The boundary of the 4-dimensional dS^4 is made of two 3-dimensional Euclidean spheres, one in the infinite past one in the future, and so the isometry of the dS^4 space-time can be said to be equivalent to the conformal symmetry group on its boundary.

In addition, in connection with MOND we know that:

4. MOND is purportedly linked with cosmology, as pointed to by relation (34) above, by which the value of a_0 is related to the geometry of the universe, in particular to the radius of the asymptotic de Sitter space-time.

5. As mentioned above (details in Milgrom 1997), the deep-MOND limit of the gravitational field equation of the Bekenstein & Milgrom (1984) formulation has the full symmetry of the conformal group in 3-dimensional Euclidean space. More specifically, the possibility was raised there that this MOND-limit of the theory could be a classical limit of some conformal (quantum) field theory (CFT). This was based on the identification of certain quantities in the theory as the "primary fields", in the jargon of CFT, which come complete with their correlation functions and anomalous dimensions. The "correlation functions", which are related to the energy of a configuration of N point masses, m_i, placed at positions $\mathbf{r}_i$, contain all the observable information about gravitational systems made of massive particles; so, knowing them is tantamount to solving the problem completely. In this theory for the gravitational field, time has disappeared, as explained above in connection with the symmetry, and has to be brought back in (in a still moot manner) when we consider the equation of motion of masses.

In light of these, one may conjecture that the MOND-cosmology connection is such that local gravitational physics would take exactly the deep-MOND form in an exact de Sitter universe[18]. This is based on the equality of the symmetry groups of dS^4 and of the MOND limit of the Bekenstein-Milgrom formulation (point 2 and 5 above), both groups being $SO(4,1)$. The fact that today we see locally a departure from the exact MOND-limit physics–i.e., that the interpolating functions have the form they have, and that a_0 is finite and serves as a transition acceleration–stems from the departure of our actual space-time from exact dS^4 geometry: The broken symmetry of our space-time is thus echoed in the broken symmetry of local physics. This conjecture would then imply that the deep-MOND limit will be approached by local physics,

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[17]Space then has to be compactified by adding the point at infinity.

[18]This conjecture differs from another, earlier one by which MOND with its two different limits would apply in an exact dS^4, with the de Sitter radius, as represented by $\hat{a}_0$, dictating the transition acceleration.

as our universe approaches asymptotically an exact dS^4 geometry in the future[19,20]. For a relation such as eq.(36) this would mean that $\mathcal{A} \to \infty$ for $a_H/\hat{a}_0 \to 1$. But, more generally, local physics would have to be able to sense, somehow, whether the local space-time neighborhood is nearly pseudo-spherical with positive curvature (i.e., nearly that of a dS^4), and adjust the value of a_0 accordingly. Local physics can sense this using only local indicators such as the derivative of the Hubble constant. For example, as one out of many possible relations, $a_0 \propto cH/(1+q)$ (H being the expansion rate, and $q = -1 - \dot{H}/H^2$ the deceleration parameter) would give the deep-MOND limit in periods of exponential growth of the cosmological scale factor (when $\dot{H} \ll H^2$), and $a_0 \sim cH \propto t^{-1}$ in periods of power-law growth; whereas $a_0 \propto \hat{a}_0/(1+q)$ would also give local deep-MOND behavior during exponential growth, but constant $a_0 \sim \hat{a}_0$ for power-law growth.

Based on point 1-3 above, a conjecture has been launched of a correspondence between conformal field theories in 3-D Euclidean space and gravity in dS^4 (starting, to my knowledge, with Strominger 2001, and Spradlin et al. 2001; see also Medved 2002, and references therein)[21]. If such a correspondence could be found in connection with MOND, it might point to a relativistic gravity theory with the symmetries of dS^4 that leads to the energy functions encapsuling the deep-MOND limit, and, more generally, show how a compound MOND theory can follow in the context of our less ideal cosmological space-time.

Whether the actual correspondence between theories in the two spaces exist or not, the symmetry connection is there and could be highly pertinent for MOND.

If indeed, as speculated here, local physics is tending to the deep-MOND limit in the asymptotically dS future, with $a_0 \to \infty$, and with convergence to 3-D conformal invariance, this invariance would also apply to the deep-MOND limit *today*; i.e., to systems that are today deep in the MOND regime as defined by today's value of a_0 (such as low surface brightness galaxies, and the outskirts of all galaxies). This is because both physical limits are formally effected by taking $a_0 \to \infty$ in the field equations, and would correspond to the same limiting theory. Also note that while the symmetry gained in the limit is conformal invariance in 3-D Euclidean space, in the context of MOND this would require that the gravitational potential has zero scaling dimension, which, in turn, implies also the space-time scale invariance of the equations of motion, our main topic here.

In any event, the relation between MOND and cosmology is presumably not one-way, with only cosmology affecting local dynamics (e.g., by dictating a value of a_0). The equations governing cosmology may well be also affected by the departure from standard physics that MOND entails; and this needs to be understood as well.

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[19] When I speak of the deep-MOND limit as it pertains to present day physics, I mean its application to systems for which all accelerations are much smaller than the present value of a_0, and whose dynamics can thus be deduced by formally taking a_0 to infinity with MGa_0 fixed. Here, I mean that the limit would actually be approached in the future, asymptotically dS^4 universe for all local systems. So, any gravitational system will go deeper and deeper into the MOND regime. One has then to consider how the structure of the system responds to the adiabatic changes involved. That masses also change, so as to keep m_i/M_0 fixed in the limit, is also a possibility to reckon with.

[20] The deep-MOND limit might then also prevail during past inflationary phases.

[21] This is the dS analog in 4-D, of the more frequently discussed, and more strongly motivated, correspondence between conformal field theories in 4-D Minkowski space-time and quantum gravity in space-times built on 5-D Anti dS space-time–the so called AdS/CFT, or Maldacena, conjecture.

5.1.   A note on extant relativistic theories

In light of what I said above, I believe that a lasting marriage of MOND with the principles of relativity will emerge only when we understand the MOND-cosmology connection. In such an eventuality, a_0 will be derived from the theory, and so will the various interpolating functions. It is then not clear that it is meaningful to ask, at present, what the deep-MOND limit will be like in relativistic MOND. Nevertheless, relativistic versions of MOND have been propounded, and one may wonder how the discussion here bears on such theories. The state of the art of these efforts is the Tensor-Vector-Scalar (TeVeS) theory formulated by Bekenstein (2004)–extending previous ideas by Sanders (1997)–hinging gravity on a vector field in addition to the usual metric tensor and a scalar field. This theory has also been recast as a special case of so called Generalized Einstein Aether theories by Zlosnik et al. (2007). Such theories introduce the limiting acceleration and the interpolating function by hand. Nevertheless, they are most valuable as intermediate steps, perhaps pointing the way to more fundamental MOND theories. Inasmuch as these theories have a nonrelativistic limit that corresponds to the modified Poisson equation discussed above, they enjoy its symmetries in the deep MOND limit of their *nonrelativistic* limit. However, As I have stressed several times in the past, because of relation (34), a system that is both in the relativistic strong field limit, and is deep in the MOND regime, must have a size much larger then the Hubble radius. Cosmology itself is only at the boundary in the MOND sense, with typical accelerations of order a_0. It is thus difficult to give a meaning to a relativistic deep-MOND limit, let alone have any observational guidance regarding such systems. Our topic here is the very deep MOND limit, and there is little of it that we can apply in a sensible way to relativistic theories per se.

I just note, in this connection, that the appearance of the speed of light, c, in a MOND based theory does not, in itself, spoil space-time scale invariance, since the value of c does not change under simultaneous scaling of the units of length-time. A relativistic theory could, however, not be invariant to scaling, and still have an invariant nonrelativistic limit, with the symmetry breaking vanishing in the limit $c^{-1} = 0$.

6.   Discussion

I note, in passing, that various hints finger space-time conformal maps (not the Euclidean space transformations discussed above) as being relevant to the various MOND theories. One hint stems from the discussion here; in particular, the fact that scaling is one of the conformal maps of space-time. Also, in Milgrom (2005) I suggested that the symmetries underlying MOND may have to do with transformations that map inertial world lines in dS space-time into constant accelerations ones (another suggestion of a MOND-dS connection). As it turns out, these are the non-rigid conformal transformations[22]. Such conformal maps may thus play a role in connection with MOND similar to that of Lorentz transformations in connection with relativity.

I finally note that the scaling symmetry of the deep-MOND limit is not forced by dimensional consideration alone, as is the case for Newtonian gravity. Starting with the latter, recall that the equations describing Newtonian, purely gravitational systems are invariant to $(t, \mathbf{r}) \rightarrow (\lambda t, \lambda^{2/3} \mathbf{r})$. This is because $m_i G$, the only dimensioned constants that appear, retain their values under the corresponding change of units[23]. General

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[22]It has long been realized that conformal transformations map inertial world lines in Minkowski space-time to accelerated ones, as discussed, for example in Fulton et al. 1962, and in references therein.

[23]This scaling property remains valid for nonrelativistic electrostatics, since the action describing a system of masses and

Relativity does not enjoy the same symmetry because $c \to \lambda^{-1/3}c$; but its limit $c^{-1} = 0$ does.

In MOND, we have two types of dimensioned constants, m_iG and a_0. So, in the limit $a_0 \to \infty$, we could have had invariance to any scaling of the form $(t, \mathbf{r}) \to (\lambda t, \lambda^\beta \mathbf{r})$. This would have lead to a_0 and masses appearing in the combination $MGa_0^{1+\alpha}$ [$\alpha = 4(\beta-1)/(2-\beta)$], since these are invariant in value under the change of units that corresponds to the above scaling. Different phenomenologies would have ensued. For example the asymptotic rotational speed around a mass M would be

$$V(r) \to (MG)^{\frac{2-\beta}{4\beta}} a_0^{\frac{3\beta-2}{4\beta}} r^{\frac{\beta-1}{\beta}} = (MGa_0^{1+\alpha}r^\alpha)^{\frac{1}{4+2\alpha}}. \tag{37}$$

For Newtonian gravity $\beta = 2/3$ ($\alpha = -1$), which gives straightforwardly Kepler's third law, and the dependence of the Kepler constant on the central mass. The requirement of asymptotically flat rotation curves–the founding axiom of MOND–dictates $\alpha = 0$, $\beta = 1$.

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